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Large Elastic Deformations of Isotropic Materials. VIII. Strain Distribution around a Hole in a Sheet

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LARGE ELASTIC DEFORMATIONS OF ISOTROPIC MATERIALS VIII. STRAIN DISTRIBUTION AROUND A HOLE IN A SHEET

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CONTENTS

The deformation produced by radial forces in a thin circular sheet of incompressible highly elastic material, isotropic in its undeformed state, containing a central circular hole, is studied theoretically, and results calculated on the basis of the theory are compared with those obtained experimentally employing a vulcanized natural rubber compound as the highly elastic material.

1. Introduction

In earlier parts, a mathematical theory of the deformation of ideal highly elastic materials which are incompressible and isotropic in their undeformed state has been formulated. The elastic properties of the material are specified in terms of a stored-energy function W which must be a function of two invariants of the strain I_1 and I_2. On the basis of this theory, the forces necessary to produce certain simple types of deformation in such a highly elastic material have been calculated without the assumption of any particular form for the dependence of W on I_1 and I_2. In other cases, it has been necessary, in order to pursue the calculations to their conclusion, to assume a specific simple form for W given by

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{1·1}$$

where C_1 and C_2 are physical constants. This form was first suggested by Mooney (1940) on semi-empirical grounds, as being suitable for the description of the elastic properties of rubber. It has already been pointed out (Rivlin 1949, and part VII) that this form is the most general that can be taken by the stored-energy function if terms of third and higher degree in the principal extensions are neglected; i.e. it is the analogue for incompressible highly elastic materials of the form derived by Murnaghan (1937) for compressible highly elastic materials.

In the present part we shall discuss the problem of the deformation produced in a thin circular sheet of the highly elastic material, containing a central circular hole, when it is stretched in a circularly symmetrical manner by radial forces lying in its plane and acting on its periphery.

The radial distribution of the displacement for various values of the extension ratio at the hole is computed on the assumption that the stored-energy function W has the form (1·1). It is found that this displacement distribution is not very sensitive to the value of C_2/C_1. The results of the calculations are compared with measurements made on a test-piece of vulcanized natural rubber, and good agreement is obtained for a value of C_2/C_1 of about 0·1. It is shown further, by calculation, that for each state of deformation of the sheet, the value of I_2 varies only slightly with radius. This has the result that any slight dependence of $\partial W/\partial I_2$ on I_2, such as is indicated by the experimental results given in part VII, would not be reflected in disagreement between the experimental results and those calculated on the basis of a stored-energy function of the form (1·1), provided C_2/C_1 is given an appropriate value for each state of deformation of the sheet.

2. The stress distribution round a circular hole

Consider a thin circular sheet of incompressible highly elastic material which is isotropic in its undeformed state, of radius a_1 and uniform thickness h, containing at its centre a hole of radius a. Let this be deformed by the application of radial surface tractions uniformly distributed along its outer edge. Under the action of these forces the radius of the hole is increased to λa, say.

We choose as reference axes a cylindrical polar system (r, θ, z) having its pole at the centre and its z-axis normal to the plane of the sheet. Then, it is evident from the symmetry of the problem that a point of the material which is at (r, θ, z) in the undeformed state will move, in the deformation, to (ρ, ϑ, ζ), where $\vartheta = \theta$. Provided that the sheet is sufficiently thin for the variation of the radial displacement over its thickness to be negligible, ρ is a function of r only and $\zeta = \lambda_3 z$, where λ_3 is also a function of r only. The material at each point of the sheet is then in a state of pure strain. The principal axes of strain are in the radial, azimuthal and normal directions at each point of the sheet and the extension ratios for these three directions will be denoted by λ_1, λ_2 and λ_3 respectively, where λ_1, λ_2 and λ_3 are functions of r only. We readily see from the geometry of the system that

$$\lambda_1 = \frac{d\rho}{dr}, \quad \lambda_2 = \frac{\rho}{r} \tag{2·1}$$

and, since the material is incompressible so that $\lambda_1 \lambda_2 \lambda_3 = 1$,

$$\lambda_3 = r \Big/ \rho\left(\frac{d\rho}{dr}\right). \tag{2·2}$$

The normal components of the stress in the radial and azimuthal directions and in the direction of the normal to the sheet will be denoted $t_{\rho\rho}$, $t_{\vartheta\vartheta}$ and $t_{\zeta\zeta}$ respectively. These are given (Rivlin 1948, equations (6·3)) by

$$t_{\rho\rho}, \quad t_{\vartheta\vartheta}, \quad t_{\zeta\zeta} = 2\left[\lambda_i^2 \frac{\partial W}{\partial I_1} - \frac{1}{\lambda_i^2}\frac{\partial W}{\partial I_2}\right] + p \quad (i = 1, 2, 3), \tag{2·3}$$

where
$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad \text{and} \quad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}. \tag{2·4}$$

W is the stored-energy function for the material and is a function of I_1 and I_2. p denotes an arbitrary hydrostatic pressure. The tangential components of the stress are, from the symmetry of the problem, zero.

Since there are no surface tractions acting on the major surfaces of the sheet, we have $t_{\zeta\zeta} = 0$, so that equations (2·3) yield

$$\left.\begin{aligned} t_{\rho\rho} &= 2(\lambda_1^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}\right) \\ \text{and}\qquad t_{\vartheta\vartheta} &= 2(\lambda_2^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_1^2\frac{\partial W}{\partial I_2}\right).\end{aligned}\right\} \tag{2·5}$$

Let T_1 and T_2 be the resultants of the stresses $t_{\rho\rho}$ and $t_{\vartheta\vartheta}$ acting over the thickness of the sheet and measured per unit length in the deformed sheet. Then, since in the deformed state the thickness of the sheet at any point is $\lambda_3 h$,

$$\left.\begin{aligned} T_1 &= 2h\lambda_3(\lambda_1^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}\right) \\ \text{and}\qquad T_2 &= 2h\lambda_3(\lambda_2^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_1^2\frac{\partial W}{\partial I_2}\right).\end{aligned}\right\} \tag{2·6}$$

The equations of equilibrium for the sheet become, since no body forces are acting,

$$\frac{d}{d\rho}(\rho T_1) = T_2, \tag{2·7}$$

the remaining two equations being automatically satisfied. This equation may be rewritten

$$\frac{d}{dr}(\rho T_1) = \frac{d\rho}{dr}T_2. \tag{2·8}$$

At the periphery of the hole, i.e. where $r = a$, $t_{\rho\rho} = 0$. From the first of equations (2·5), this yields $\lambda_1 = \lambda_3$, which, with the incompressibility condition $\lambda_1\lambda_2\lambda_3 = 1$, gives

$$[\lambda_1]_{r=a} = [\lambda_3]_{r=a} = \lambda^{-\frac{1}{2}}\,(\text{say}) \quad \text{and} \quad [\lambda_2]_{r=a} = \lambda. \tag{2·9}$$

Introducing this result into the second of equations (2·6) we obtain

$$[T_2]_{r=a} = 2h\lambda^{-\frac{1}{2}}\left(\lambda^2-\frac{1}{\lambda}\right)\left[\left(\frac{\partial W}{\partial I_1}\right)_{r=a}+\frac{1}{\lambda}\left(\frac{\partial W}{\partial I_2}\right)_{r=a}\right], \tag{2·10}$$

where, from (2·9) and (2·4),

$$[I_1]_{r=a} = \lambda^2+\frac{2}{\lambda} \quad \text{and} \quad [I_2]_{r=a} = \frac{1}{\lambda^2}+2\lambda. \tag{2·11}$$

It should be noted that since $t_{\rho\rho} = t_{\zeta\zeta} = 0$ at the edge of the hole, the material of the sheet there is in simple extension.

3. Approximate solutions of the equations

By Taylor's theorem, we can express the value of ρ for any value of r thus:

$$\rho = \sum_0^\infty \frac{1}{n!}(r-a)^n\left[\frac{d^n\rho}{dr^n}\right]_{r=a}. \tag{3·1}$$

Successive approximations to ρ can be found by calculating the values of successive derivatives of ρ when $r = a$. We have, from (2·9) and (2·1), $\rho = \lambda a$ and $d\rho/dr = \lambda^{-\frac{1}{2}}$, so that

$$\rho = \lambda a+(r-a)\,\lambda^{-\frac{1}{2}} \tag{3·2}$$

provides a first approximation to the solution of the problem. For the next approximation we must calculate $d^2\rho/dr^2$.

From equation (2·8), we obtain

$$\rho\frac{dT_1}{dr}+\frac{d\rho}{dr}(T_1-T_2)=0. \tag{3·3}$$

When $r=a$, $T_1=0$, $\rho=\lambda a$ and $d\rho/dr=\lambda^{-\frac{1}{2}}$ and T_2 is given by (2·10) so that

$$\left[\frac{dT_1}{dr}\right]_{r=a}=\frac{2h}{a}\left(1-\frac{1}{\lambda^3}\right)\left[\left(\frac{\partial W}{\partial I_1}\right)_{r=a}+\frac{1}{\lambda}\left(\frac{\partial W}{\partial I_2}\right)_{r=a}\right]. \tag{3·4}$$

Now, from the first of equations (2·6), we have

$$\frac{dT_1}{dr}=T_1\left[\frac{1}{\lambda_3}\frac{d\lambda_3}{dr}+\frac{2}{\lambda_1^2-\lambda_3^2}\left(\lambda_1\frac{d\lambda_1}{dr}-\lambda_3\frac{d\lambda_3}{dr}\right)+\frac{1}{\dfrac{\partial W}{\partial I_1}+\lambda_2^2\dfrac{\partial W}{\partial I_2}}\frac{d}{dr}\left(\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}\right)\right], \tag{3·5}$$

where T_1 is given by the first of equations (2·6). Thus

$$\left[\frac{dT_1}{dr}\right]_{r=a}=\frac{4h}{\lambda}\left[\left(\frac{d\lambda_1}{dr}-\frac{d\lambda_3}{dr}\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)\right]_{r=a}. \tag{3·6}$$

In view of the incompressibility condition $\lambda_1\lambda_2\lambda_3=1$, we have

$$\frac{1}{\lambda_1}\frac{d\lambda_1}{dr}+\frac{1}{\lambda_2}\frac{d\lambda_2}{dr}+\frac{1}{\lambda_3}\frac{d\lambda_3}{dr}=0, \tag{3·7}$$

whence
$$\left[\frac{d\lambda_1}{dr}+\frac{d\lambda_3}{dr}\right]_{r=a}=-\lambda^{-\frac{3}{2}}\left[\frac{d\lambda_2}{dr}\right]_{r=a}. \tag{3·8}$$

From (2·1)
$$\frac{d\lambda_2}{dr}=\frac{1}{r}\frac{d\rho}{dr}-\frac{\rho}{r^2}, \tag{3·9}$$

so that
$$\left[\frac{d\lambda_2}{dr}\right]_{r=a}=\frac{1}{a}(\lambda^{-\frac{1}{2}}-\lambda). \tag{3·10}$$

From (3·8), (3·10), (3·6) and (3·4), we obtain

$$\left[\frac{d\lambda_1}{dr}\right]_{r=a}=\frac{\lambda\left[\left(\dfrac{\partial W}{\partial I_1}\right)_{r=a}\left(1-\dfrac{3}{\lambda^3}+\dfrac{2}{\lambda^{\frac{3}{2}}}\right)+\dfrac{1}{\lambda}\left(\dfrac{\partial W}{\partial I_2}\right)_{r=a}\left(2\lambda^{\frac{3}{2}}-1-\dfrac{1}{\lambda^3}\right)\right]}{4a\left[\left(\dfrac{\partial W}{\partial I_1}\right)_{r=a}+\lambda^2\left(\dfrac{\partial W}{\partial I_2}\right)_{r=a}\right]}. \tag{3·11}$$

Now, since $d\lambda_1/dr=\rho_{rr}$, we can readily obtain the second approximation to ρ from (3·1).

We can obtain a first approximation to the dependence of ρ on r, in the neighbourhood of the hole, which is somewhat more accurate than (3·2), in the following manner. Since $t_{\rho\rho}=0$, when $r=a$, we have from the first of equations (2·5), (2·1) and (2·2)

$$\rho_r=(r/\rho)^{\frac{1}{2}}. \tag{3·12}$$

This yields with the condition that $\rho=\lambda a$, when $r=a$,

$$\rho^{\frac{3}{2}}=r^{\frac{3}{2}}+a^{\frac{3}{2}}(\lambda^{\frac{3}{2}}-1). \tag{3·13}$$

For this result to be accurately applicable, we would require that $t_{\rho\rho}=0$ over its whole range of applicability. It will, however, be approximately valid so long as $t_{\rho\rho}$ is small compared with $t_{\vartheta\vartheta}$.

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The solution of the problem in the limiting case of infinitesimally small deformations can readily be obtained from classical elasticity theory. This yields

$$\frac{\rho}{r}-1=\tfrac{1}{4}(\lambda-1)\left(1+\frac{3a^2}{r^2}\right). \tag{3·14}$$

The equations given in § 2 can, of course, be solved, with appropriate approximations, to give the same result.

4. Numerical solution of the problem

The problem may be readily solved numerically, if the form of W as a function of I_1 and I_2 is known. Let us suppose that when $r=r'$, we know the values of λ_1 and λ_2. Then, we can calculate their values for $r=r'+\Delta r$, where Δr is sufficiently small, in the following manner.

Equation (3·9) may be rewritten

$$\frac{d\lambda_2}{dr}=\frac{1}{r}(\lambda_1-\lambda_2). \tag{4·1}$$

Whence
$$[\lambda_2]_{r=r'+\Delta r}=[\lambda_2]_{r=r'}+\frac{\Delta r}{r'}[\lambda_1-\lambda_2]_{r=r'}. \tag{4·2}$$

If the form of W as a function of I_1 and I_2 is known, we can calculate $[T_1]_{r=r'}$ and $[T_2]_{r=r'}$ from equations (2·6), λ_3 being obtained from the relation $\lambda_1\lambda_2\lambda_3=1$ and I_1 and I_2 from equations (2·4). Equation (3·3) may be rewritten

$$\frac{dT_1}{dr}=\frac{\lambda_1}{\lambda_2 r}(T_2-T_1). \tag{4·3}$$

From (2·4) and (3·7) we see that

$$\frac{d}{dr}\left(\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}\right)=2\left[A\frac{1}{\lambda_1}\frac{d\lambda_1}{dr}(\lambda_1^2-\lambda_3^2)+B\frac{1}{\lambda_2}\frac{d\lambda_2}{dr}(\lambda_2^2-\lambda_3^2)+\lambda_2\frac{d\lambda_2}{dr}\frac{\partial W}{\partial I_2}\right], \tag{4·4}$$

where
$$A=\frac{\partial^2 W}{\partial I_1^2}+2\lambda_2^2\frac{\partial^2 W}{\partial I_1\partial I_2}+\lambda_2^4\frac{\partial^2 W}{\partial I_2^2}$$
and
$$B=\frac{\partial^2 W}{\partial I_1^2}+(\lambda_1^2+\lambda_2^2)\frac{\partial^2 W}{\partial I_1\partial I_2}+\lambda_1^2\lambda_2^2\frac{\partial^2 W}{\partial I_2^2}. \tag{4·5}$$

Employing the relations (3·7) and (4·4) in (3·5) and substituting for T_1 from (2·6), we obtain

$$\frac{d\lambda_1}{dr}=\frac{1}{Q}\left(\frac{1}{2h}\frac{dT_1}{dr}-P\right), \tag{4·6}$$

where
$$P=\frac{\lambda_3}{\lambda_2}\frac{d\lambda_2}{dr}\left[(3\lambda_3^2-\lambda_1^2)\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}(\lambda_1^2+\lambda_3^2)+2(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)B\right]$$
and
$$Q=\frac{\lambda_3}{\lambda_1}\left[(3\lambda_3^2+\lambda_1^2)\left(\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}\right)+2A(\lambda_1^2-\lambda_3^2)^2\right]. \tag{4·7}$$

From the value of $d\lambda_1/dr$ calculated by (4·6) for $r=r'$ we can find $[\lambda_1]_{r=r'+\Delta r}$ by the relation

$$[\lambda_1]_{r=r'+\Delta r}=[\lambda_1]_{r=r'}+\left[\frac{d\lambda_1}{dr}\right]_{r=r'}\Delta r. \tag{4·8}$$

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We thus see that if λ_1 and λ_2 are known for $r = r'$, they may be calculated for $r = r' + \Delta r$. Since for $r = a$, $\lambda_1 = \lambda^{-\frac{1}{2}}$ and $\lambda_2 = \lambda$, we can calculate λ_1 and λ_2 for all values of r by successive application of the method outlined.

If the stored-energy function W has the simple form

$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \tag{4·9}$$

where C_1 and C_2 are constants, then from (2·6), T'_1 and T'_2, defined by

$$T'_1 = T_1/2hC_1 \quad \text{and} \quad T'_2 = T_2/2hC_1, \tag{4·10}$$

are given by

$$\left.\begin{aligned} T'_1 &= \lambda_3(\lambda_1^2 - \lambda_3^2)\,(1 + \alpha\lambda_2^2) \\ T'_2 &= \lambda_3(\lambda_2^2 - \lambda_3^2)\,(1 + \alpha\lambda_1^2), \end{aligned}\right\} \tag{4·11}$$

and

where $\alpha = C_2/C_1$.

Equation (4·6) may now be written

$$\frac{d\lambda_1}{dr} = \frac{\lambda_1 \dfrac{dT'_1}{dr} - \dfrac{1}{\lambda_2^2}\dfrac{d\lambda_2}{dr}\left[(3\lambda_3^2 - \lambda_1^2) + \alpha\lambda_2^2(\lambda_1^2 + \lambda_3^2)\right]}{\lambda_3(3\lambda_3^2 + \lambda_1^2)\,(1 + \alpha\lambda_2^2)}. \tag{4·12}$$

5. Experimental arrangement

In order to compare the theoretical predictions of the previous sections with experimental results, a circular sheet of vulcanized natural rubber, about 5 in. in diameter and $\frac{1}{16}$ in. thick, was employed. The mix from which the rubber sheet was vulcanized had the following composition in parts by weight: natural rubber (smoked sheet) 100, sulphur 2, zinc oxide 2, accelerator (M.B.T.S.) 1, stearic acid 0·5 and nonox 0·5. Vulcanization was effected by raising the temperature slowly to 141° C over a period of 30 min. and then maintaining the temperature at 141° C for a further 30 min. A centrally placed hole about 1 in. in diameter was cut in the sheet. A number of diameters and concentric circles were drawn on the surface of the sheet in ink, and sixteen circular holes, $\frac{1}{8}$ in. in diameter, were bored in the sheet near its circumference and at regular angular intervals.

The sheet was supported from hooks uniformly spaced on the circumference of a circle by means of strings attached to it through these holes. It was arranged that the lengths of these strings could be varied by means of small runners. By suitably adjusting the lengths of these strings, the sheet was stretched in its own plane in such a way that the circles marked on it remained circles and the diameters remained diameters and were not rotated. This was not, of course, possible in the region near the periphery of the sheet where irregularities were caused by the holes through which the strings were attached. However, these irregularities became negligible within $\frac{1}{2}$ in. of the holes measured radially on the sheet, and measurements were made only on the central portion of the sheet where they could be neglected.

The radii r of the circles and the radius a of the central hole were first measured with the sheet in its undeformed state. Then, the sheet was stretched to various extents, in the manner described, and in each state of stretch the radii ρ of the circles and the radius λa of the central hole were measured. All of these measurements were carried out with a travelling microscope fitted with a vernier which could be read to 0·02 mm.

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6. Experimental and theoretical results

In figure 1, curves I to IV show the relations between ρ/r and r/a for $\lambda = 2{\cdot}10$, $3{\cdot}18$, $4{\cdot}67$ and $6{\cdot}01$ respectively, calculated in the manner described in §4, employing a value for α of $0{\cdot}1$. It is seen that the experimental results, denoted by circles, show good agreement with the theoretical curves. In the case when $\lambda = 6{\cdot}01$, which will be discussed later, the agreement is less close than in the other cases.

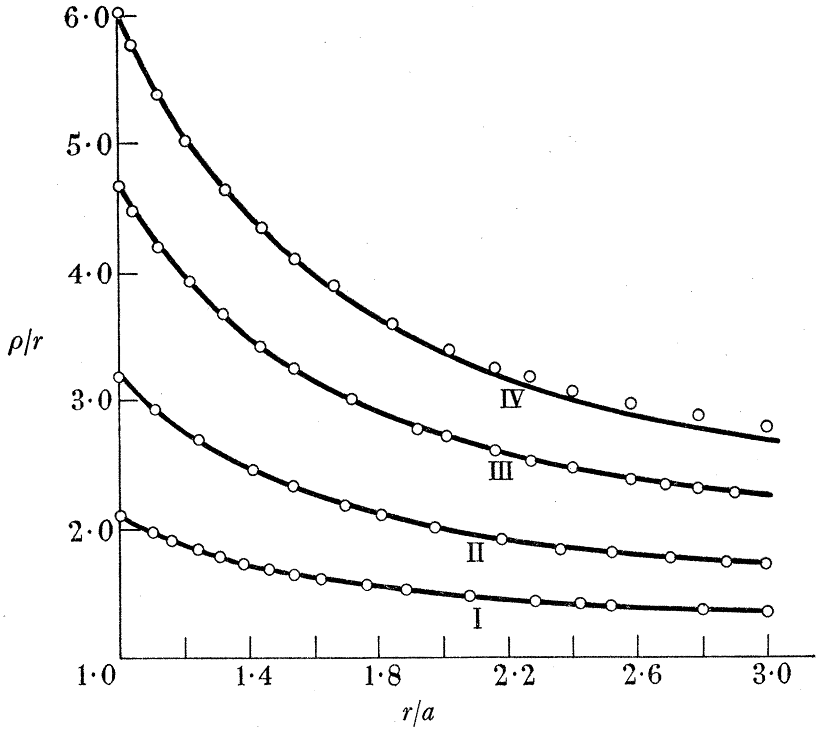

Figure 1. Calculated plot of ρ/r against r/a for $\alpha = 0{\cdot}1$ compared with experimental results for various values of λ.

In figure 2, curves I, II and III show the relations between ρ/r and r/a calculated for $\lambda = 2{\cdot}10$ and values for α of 0, $0{\cdot}1$ and $0{\cdot}2$ respectively. Curve IV represents the asymptotic formula (3·13). In figure 3, curves I to IV show the relations between ρ/r and r/a calculated for $\lambda = 6{\cdot}01$ and $\alpha = 0$, $0{\cdot}08$, $0{\cdot}1$ and $0{\cdot}2$ respectively. Curve V represents the asymptotic formula (3·13). It is seen that the experimental results for this value of λ agree well with curve II for which $\alpha = 0{\cdot}08$. For all the values of λ, the calculated results are not very sensitive to the value of α chosen.

In carrying out the calculations, the value taken for Δr varied from 0·01 to 0·05, the larger values being taken, in general, for the higher values of r/a where the variation of ρ/r with r/a is less rapid. The absolute errors in the computation increase, of course, with r/a and at $r/a = 3$ are estimated at less than 0·04 in the case when $\lambda = 6{\cdot}01$, where they are greatest.

In carrying out the calculations, it has been assumed that W has the Mooney form (4·9), i.e. $\partial W/\partial I_1$ and $\partial W/\partial I_2$ are constants. In view of the results obtained in the preceding part (VII), with a similar type of vulcanized rubber, indicating a dependence of $\partial W/\partial I_2$ on I_2, it may at first sight appear strange that such good agreement should be obtained between theory and experiment by taking $\partial W/\partial I_2$ constant. This may to some extent be accounted for by the insensitivity of the results to the value of α employed in the calculation. Further, the values of I_1 and I_2 at each point of the deformed sheet, for a given value of λ and α, can be calculated by means of equations (2·4). Results of such calculations are shown in figures 4 and 5 for

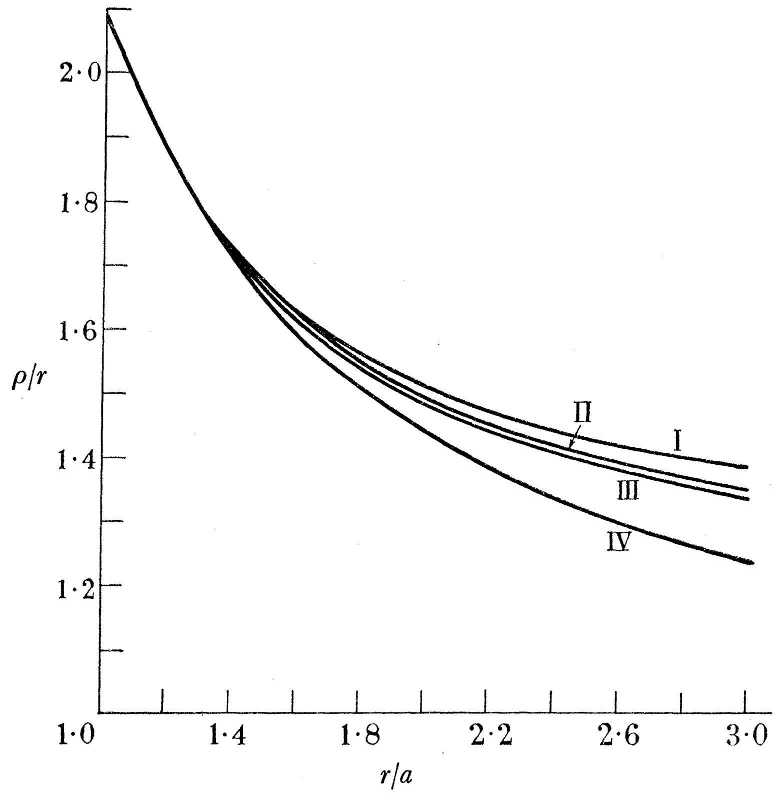

FIGURE 2. Calculated plot of ρ/r against r/a for $\lambda = 2{\cdot}10$ and various values of α. I, $\alpha = 0$; II, $\alpha = 0{\cdot}1$; III, $\alpha = 0{\cdot}2$; IV, asymptotic formula.

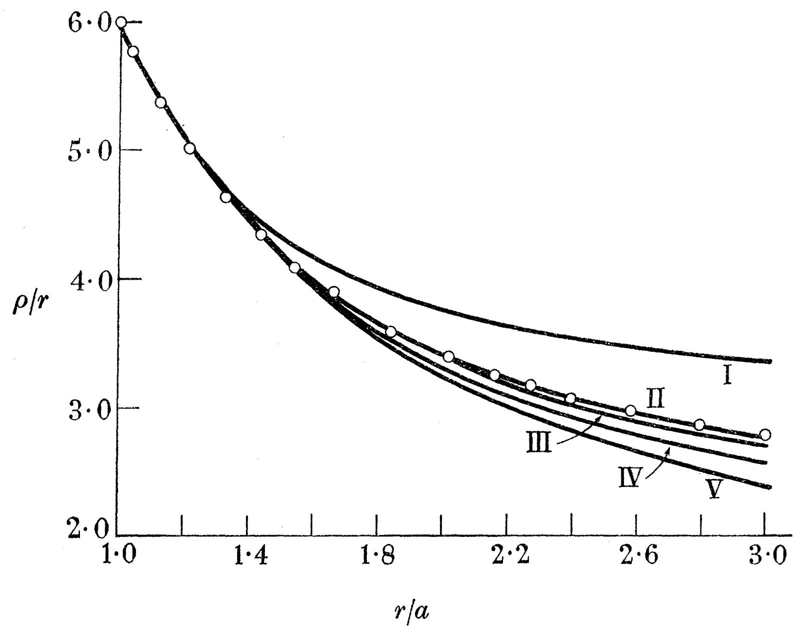

FIGURE 3. Calculated plot of ρ/r against r/a for $\lambda = 6{\cdot}01$ and various values of α compared with experimental results. I, $\alpha = 0$; II, $\alpha = 0{\cdot}08$; III, $\alpha = 0{\cdot}1$; IV, $\alpha = 0{\cdot}2$; V, asymptotic formula.

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$\alpha = 0{\cdot}1$ and the values of λ obtaining in the experiments. It is seen that I_2 is approximately independent of r/a for each value of λ. Thus, even if formally $\partial W/\partial I_2$ depends to some extent on I_2, its value on each curve will be substantially constant. The values of α for which the calculated curves agree with the experimental results are approximately equal to the values found in the foregoing paper for $(\partial W/\partial I_2)/(\partial W/\partial I_1)$ at the appropriate values of I_2. The

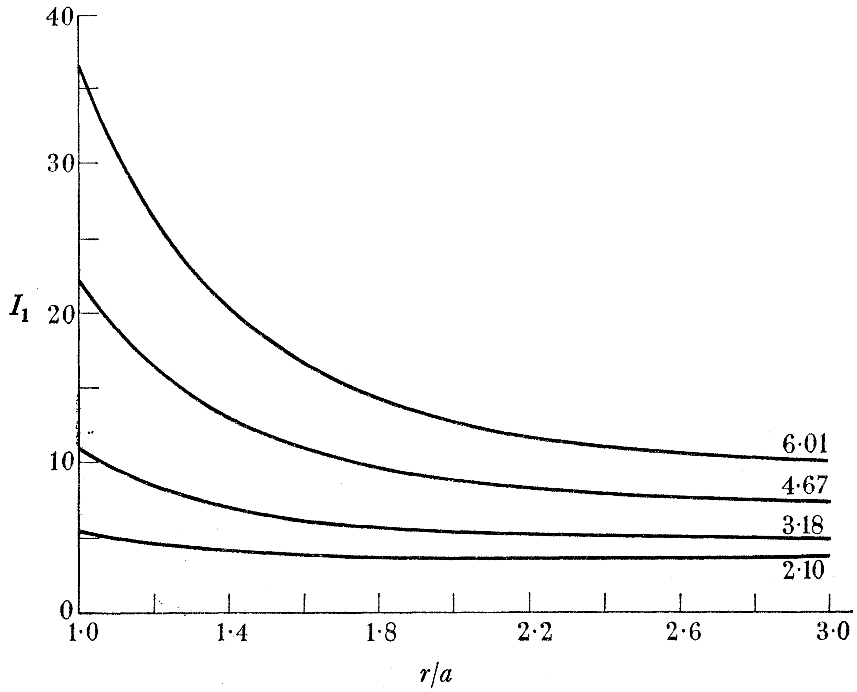

FIGURE 4. Calculated radial variation of I_1 for $\alpha = 0{\cdot}1$ and various values of λ indicated on the curves.

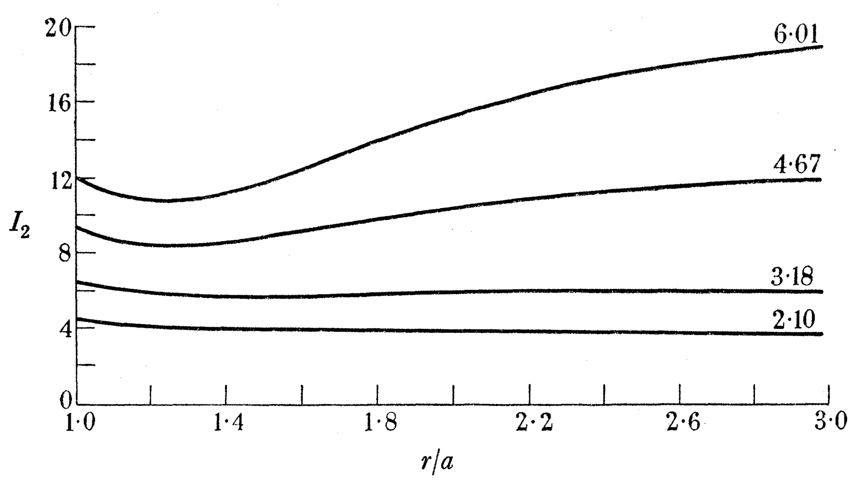

FIGURE 5. Calculated radial variation of I_2 for $\alpha = 0{\cdot}1$ and various values of λ.

insensitivity of the shape of the ρ/r against r/a curves to dependence of $\partial W/\partial I_2$ on I_2 is the more marked since I_2 varies most rapidly with r/a near the hole, and it has already been seen that in this region the relation between ρ/r and r/a is determined chiefly by the asymptotic formula (3·13) which is independent of the form of the stored-energy function. This probably accounts also for the fact that the agreement between theory and experiment is not vitiated by the crystallization of the rubber in the immediate neighbourhood of the hole resulting from its state of great extension.

The slight fall in the value of α, for which agreement is obtained between the experimental and theoretical results, occurring at the highest value considered of λ—and therefore of I_2—is in qualitative agreement with the fall of $\partial W/\partial I_2$ with increase in I_2 indicated in part VII. It should be borne in mind, however, that the results obtained for the lower values of λ could be very nearly equally well fitted if a value of α of 0·08 were taken in carrying out the calculations.

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